Elements and Applications of p-adic Analysis

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Structure of the Presentation

1. About $\mathbb{Q}_p$

2. Applications of $\mathbb{Q}_p$

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A Motivating Example: $1 + 2 + 4 + 8 + \cdots = -1$

Recall the geometric series:

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Question: Can we make sense of this?

Yes! (In $\mathbb{Q}_2$)

What is $\mathbb{Q}_p$?

Definition: $|\cdot|_p$

Let $\frac{a}{b} \in \mathbb{Q}^\times$ and p a (positive) prime integer. Then $\frac{a}{b} = p^n \frac{a'}{b'}$ with $\gcd(p, a') = \gcd(p, b') = 1$. Define

$$\left|\frac{a}{b}\right|_p = p^{-n}.$$

Extend to $\mathbb{Q}$ by $\left|0\right|_p = 0$.

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Definition: $\mathbb{Q}_p$

$\mathbb{Q}_p$ is the Cauchy completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

The Return of $1 + 2 + 4 + 8 + \cdots = -1$

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Convergence!

Let $S_n = \sum_{k=0}^{n} p^k$. For $m > n$ sufficiently large:

$$\begin{aligned} \left|S_m - S_n\right|_p &= \Big|\sum_{k=n+1}^{m} p^k\Big|_p \\ &= \left|p^{n+1}(1 + p + \cdots + p^{m-(n+1)})\right|_p \\ &= p^{-(n+1)} < \varepsilon. \end{aligned}$$

Some Aspects of the (Ultra)Metric Structure of $\mathbb{Q}_p$

- $|\cdot|_p$ satisfies the strong triangle inequality: $|a+b|_p \leq \max(|a|_p, |b|_p)$

Example

For $a, b \in \{1, \ldots, p-1\}$ and $m \geq n$ integers:

$$|p^m a + p^n b|_p = |p^n(p^{m-n}a + b)|_p \leq p^{-n}$$

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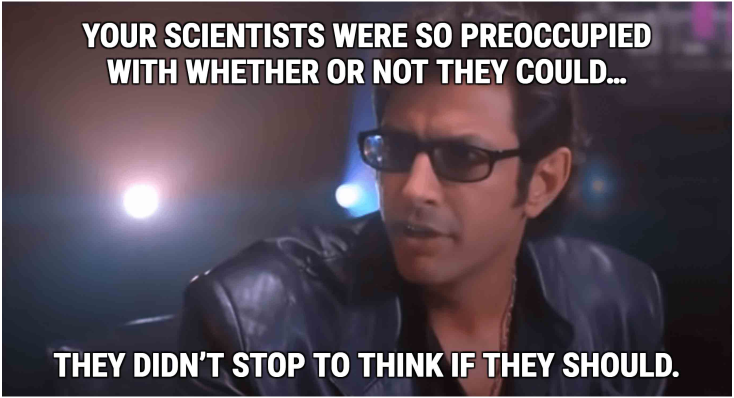
- $|\cdot|_p$ is discretely valued except around 0: $\mathrm{im}|\cdot|_p = \{p^k : k \in \mathbb{Z}\} \cup \{0\}$
- Balls in $\mathbb{Q}_p$ only intersect trivially, and every point inside a ball is at its center
- $\mathbb{Q}_p$ is totally disconnected: the only nonempty connected subsets are singletons
- $\mathbb{Q}_p$ is "Cantor set"-like: picture a p-regular tree with infinite "depth" and finite but unbounded "height"

Structure of the Presentation

Some Areas of Application



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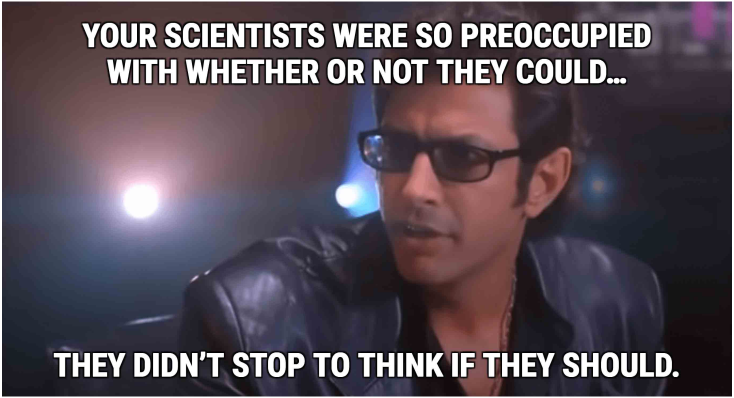

- Planck-scale physics and the Volovich hypothesis
- Local-global (Hasse-Minkowski) Principle and number field invariance
- Hierarchical structures and complex systems
- p-adic stochastic processes

A Recent Result on a p-adic Stochastic Process

Real Brownian Motion/Diffusion

- An $\mathbb{R}^d$-valued Brownian Motion is modeled by a stochastic process $\vec{W}_t$ whose law is the fundamental solution to the d-dimensional heat equation.

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p-adic Brownian Motion/Diffusion (Varadarajan, 1997)

The pseudo-differential heat equation on $\mathbb{Q}_p^d$ (with the max-norm) defines a $\mathbb{Q}_p^d$-valued analogue, $\vec{X}_t$, of the $\mathbb{R}^d$-valued stochastic process $\vec{W}_t$

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Theorem (R.R. & D. Weisbart, 2022)

A $\mathbb{Q}_p^d$-valued Brownian Motion is a d-vector of dependent $\mathbb{Q}_p$-valued Brownian Motions

Conclusions

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